New Approach to Nonperturbative Quantum Mechanics with Minimal Length Uncertainty

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(Dated: January 16, 2012)

The existence of a minimal measurable length is a common feature of various approaches to quantum gravity such as string theory, loop quantum gravity and black-hole physics. In this scenario, all commutation relations are modified and the Heisenberg uncertainty principle is changed to the so-called Generalized (Gravitational) Uncertainty Principle (GUP). Here, we present a one-dimensional nonperturbative approach to quantum mechanics with minimal length uncertainty relation which implies $X = x$ to all orders and $P = p + \frac{1}{3}\beta p^3$ to first order of GUP parameter β, where X and P are the generalized position and momentum operators and $[x, p] = i\hbar$. We show that this formalism is an equivalent representation of the seminal proposal by Kempf, Mangano, and Mann and predicts the same physics. However, this proposal reveals many significant aspects of the generalized uncertainty principle in a simple and comprehensive form and the existence of a maximal canonical momentum is manifest through this representation. The problems of the free particle and the harmonic oscillator are exactly solved in this GUP framework and the effects of GUP on the thermodynamics of these systems are also presented. Although X, P, and the Hamiltonian of the harmonic oscillator all are formally self-adjoint, the careful study of the domains of these operators shows that only the momentum operator remains self-adjoint in the presence of the minimal length uncertainty. We finally discuss the difficulties with the definition of potentials with infinitely sharp boundaries.

PACS numbers: 04.60.Bc

I. INTRODUCTION

The unification of general relativity with the laws of quantum mechanics is one of the oldest wishes of theoretical physicists from the birth of quantum mechanics. We can mention the canonical quantization [1] and the path integral quantization of gravity [2] as two well-known but old proposals which tried to present a quantization scheme for gravity. However, from the field theoretical viewpoint, the theory of relativity is not renormalizable and leads to ultraviolet divergencies. Moreover, around the Planck energy scale, the effects of gravity are so important that they would result in discreteness of the spacetime manifold. This argument is based on the fact that, when we try to probe small distances with high energies, it will significantly disturb the spacetime structure by the gravitational effects. However, the theory can be renormalizable by introducing a minimal observable length as an effective cutoff in the ultraviolet domain.

The existence of a minimum measurable length is one of the common aspects of various candidates of quantum gravity such as string theory, loop quantum gravity, and quantum geometry. Within a string-theoretical argument, we can say that a string cannot probe distances smaller than its length. Moreover, some Gedanken experiments in black-hole physics and noncommutativity of the spacetime manifold all agree on the existence of a minimal observable distance of the order of the Planck length $\ell_{Pl} = \sqrt{G\hbar/c^3} \approx 10^{-35}m$, where G is Newton's constant [3–7]. In fact, the finite resolution of spacetime points is a consequence of finite time measurement. In principle, one can probe very short distances in $D0$-branes but in an infinite time.

Note that, this is in obvious contradiction with the Heisenberg Uncertainty Principle (HUP) which puts no lower or upper bound on the nonsimultaneous measurement of the position or the momentum of a particle. In fact, in ordinary quantum mechanics ΔX can be made arbitrarily small by letting ΔP to grow correspondingly. However, for energies close to the Planck energy, the particle's Schwarzschild radius and its Compton wavelength become approximately in the order of the Planck length. So, in order to merge the idea of the minimal length into quantum mechanics, we need to modify the ordinary uncertainty principle to the so-called Generalized Uncertainty Principle (GUP). Indeed, the notion of minimal length should quantum mechanically be described as a minimal uncertainty in position measurements. The introduction of this idea has drawn much attention in recent years and many papers have been appeared in the literature to address the effects of GUP on various quantum mechanical systems and phenomena [8–24].

In this paper, we present a nonperturbative approach to one-dimensional gravitational quantum mechanics which implies a minimal length uncertainty so that the generalized position operator does not change to all orders, that is, $X = x$ and the generalized momentum operator is given by $P = p + \frac{1}{3}\beta p^3$ to first order of the GUP parameter. In this formalism the generalized position and momentum operators satisfy $[X, P] = i\hbar(1 + \beta P^2)$ where x and p are the ordinary position and momentum operators $[x, p] = i\hbar$. We show that this proposal is equivalent

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with Kempf, Mangano, and Mann (KMM) representation, but it only modifies the kinetic part of the Hamiltonian and has no effect on the potential part. Moreover, this representation agrees with perturbative approaches and predicts the presence of a maximal canonical momentum p_{max}. Here, we consider the problems of the free particle and the harmonic oscillator in the context of the generalized uncertainty principle and obtain the exact eigenvalues and corresponding eigenfunctions. Then, we discuss the consequences of the minimal uncertainty in position measurement on the partition function, mean energy, and heat capacity of these systems. The difficulties with potentials with infinitely sharp boundaries are also presented.

II. THE GENERALIZED UNCERTAINTY PRINCIPLE

According to the Heisenberg uncertainty relation, in principle, we can separately measure the position and momentum of particles with arbitrary precision. Thus, if there is a genuine lower bound on the results of the measurements, the Heisenberg uncertainty relation should be modified. Here we consider a generalized uncertainty principle which results in a minimum observable length

$$\Delta X \Delta P \geq \frac{\hbar}{2}\left(1+\beta(\Delta P)^2+\zeta\right), \tag{1}$$

where β is the GUP parameter and ζ is a positive constant that depends on the expectation value of the momentum operator. We also have $\beta=\beta_0/(M_{Pl}c)^2$ where M_{Pl} is the Planck mass and β_0 is of the order of one. Note that the deviation from the Heisenberg picture takes place in the high energy limit where the quantum gravity effects are dominant. So, for the energies much smaller than the Planck energy $M_{Pl}c^2\sim 10^{19}$ GeV, we should recover the famous Heisenberg uncertainty relation. It is straightforward to check that the above inequality relation (1) implies the existence of an absolute minimal length uncertainty as $(\Delta X)_{min}=\hbar\sqrt{\beta}$. In the context of string theory, we can interpret this length as the string length. Accordingly, the string's length is proportional to the square root of the GUP parameter. In one-dimension, the above uncertainty relation can be obtained from a deformed commutation relation, namely

$$[X,P]=i\hbar(1+\beta P^2), \tag{2}$$

where, for $\beta=0$ we recover the well-known commutation relation in ordinary quantum mechanics. Now using Eqs. (1) and (2) we can find the relation between ζ and the expectation value of the momentum operator i.e. $\zeta=\beta\langle P\rangle^2$. As Kempf, Mangano, and Mann have suggested in their seminal paper, in momentum space representation, we can write X and P as [25]

$$P\phi(p)=p\phi(p), \tag{3}$$
$$X\phi(p)=i\hbar\left(1+\beta p^2\right)\partial_p\phi(p), \tag{4}$$

where X and P are symmetric operators on the dense domain S_∞ with respect to the following scalar product

$$\langle\psi|\phi\rangle=\int_{-\infty}^{+\infty}\frac{\mathrm{d}p}{1+\beta p^2}\psi^*(p)\phi(p), \tag{5}$$

where $\int_{-\infty}^{+\infty}\frac{\mathrm{d}p}{1+\beta p^2}|p\rangle\langle p|=1$ and $\langle p|p'\rangle=\left(1+\beta p^2\right)\delta(p-p')$. In this representation the position operator is merely symmetric, but P is self-adjoint [25]. With this definition, the commutation relation (2) is exactly satisfied. Also, in quasiposition representation this formulation results in [25]

$$P\psi(x)=\frac{\tan\left(-i\hbar\sqrt{\beta}\partial_x\right)}{\sqrt{\beta}}\psi(x), \tag{6}$$
$$X\psi(x)=\left(x+\beta\frac{\tan\left(-i\hbar\sqrt{\beta}\partial_x\right)}{\sqrt{\beta}}\right)\psi(x). \tag{7}$$

Note that, for the general potential, expressing the position operator as a combination of ordinary position and momentum operators results in a complicated high-order generalized Schrödinger equation. So, finding the solutions even for the simple potentials would not be an easy task.

To overcome this problem, we propose the following generalized position and momentum operators

$$X=x, \tag{8}$$
$$P=\frac{\tan\left(\sqrt{\beta}p\right)}{\sqrt{\beta}}, \tag{9}$$

where x and p obey the canonical commutation relation $[x,p]=i\hbar$. X and P are symmetric operators on the dense domain S_∞ of functions decaying faster than any power

$$(\langle\psi|X)|\phi\rangle=\langle\psi|(X|\phi\rangle)\ \text{ and }\ (\langle\psi|P)|\phi\rangle=\langle\psi|(P|\phi\rangle), \tag{10}$$

but now with respect to the scalar product:

$$\langle\psi|\phi\rangle=\int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}}dp\,\psi^*(p)\phi(p), \tag{11}$$

The symmetry of P (9) is obvious. The symmetry of X (8) can be seen by performing partial integrations

$$\begin{aligned}&\int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}}dp\,\psi^*(p)\left(i\hbar\frac{\partial}{\partial p}\right)\phi(p)\\&=\int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}}\mathrm{d}p\,\left(i\hbar\frac{\partial\psi(p)}{\partial p}\right)^*\phi(p),\end{aligned} \tag{12}$$

which is valid for the functions vanishing at $\pm\frac{\pi}{2\sqrt{\beta}}$. Indeed, the symmetry property of the position and momentum operators ensures that all expectation values are real. This definition exactly satisfies the condition $[X,P]=i\hbar(1+\beta P^2)$ and agrees with the well-known relations [26], namely

$$X=x, \tag{13}$$
$$P=p\left(1+\frac{1}{3}\beta\,p^2\right), \tag{14}$$

to the first order of the GUP parameter. Note that to $\mathcal{O}(\beta)$, the definitions (6) and (7) result in $X = x + \beta p$ and $P = p\left(1 + \frac{1}{3}\beta\, p^2\right)$ which differ with (13) and (14).

Now, we show that our proposal and KMM representation are equivalent in essence. Indeed, they are related by the following canonical transformation:

$$X \to \left[1 + \arctan^2\left(\sqrt{\beta}P\right)\right] X, \tag{15}$$

$$P \to \arctan\left(\sqrt{\beta}P\right)/\sqrt{\beta}, \tag{16}$$

which transforms (8) and (9) into (3) and (4) subjected to condition (2). We can interpret P and p as follows: p is the momentum operator at low energies ($p = -i\hbar\partial/\partial x$) while P is the momentum operator at high energies. Obviously, this procedure affects all Hamiltonians in adopted quantum mechanics.

Note that for an operator A which is "formally" self-adjoint ($A = A^\dagger$) such as (8) and (9), this does not prove that A is truly self-adjoint because in general the domains $\mathcal{D}(A)$ and $\mathcal{D}(A^\dagger)$ may be different. The operator A with dense domain $\mathcal{D}(A)$ is said to be self-adjoint if $\mathcal{D}(A) = \mathcal{D}(A^\dagger)$ and $A = A^\dagger$. For instance, similar to KMM representation, X is merely symmetric but not self-adjoint. To see this note that in this representation and in the momentum space the wave function $\phi(p)$ have to vanish at the end of the p interval ($-\pi/2\sqrt{\beta} < p < \pi/2\sqrt{\beta}$), because the tangent function diverges there. So, X is a derivative operator $i\hbar\partial/\partial p$ on an interval with Dirichlet boundary conditions. But this means that X cannot be self-adjoint because all candidates for the eigenfunctions of X, (the plane waves, which are even normalizable) are not in the domain of X because they do not obey Dirichlet boundary conditions. Calculating the domain of the adjoint of X shows that it is larger than that of X, so X is indeed not self-adjoint i.e.

$$\begin{aligned}
&\int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \mathrm{d}p\, \psi^*(p)\left(i\hbar\frac{\partial}{\partial p}\right)\phi(p) \\
&= \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \mathrm{d}p\, \left(i\hbar\frac{\partial\psi(p)}{\partial p}\right)^* \phi(p) \\
&\quad + i\hbar\,\psi^*(p)\phi(p)\Big|_{p=+\frac{\pi}{2\sqrt{\beta}}} - i\hbar\,\psi^*(p)\phi(p)\Big|_{p=-\frac{\pi}{2\sqrt{\beta}}}.
\end{aligned} \tag{17}$$

Now since $\phi(p)$ vanishes at $p = \pm\frac{\pi}{2\sqrt{\beta}}$, $\psi^*(p)$ can take any arbitrary value at the boundaries. The above equation implies that X is symmetric, but it is not a self-adjoint operator. Although its adjoint $X^\dagger = i\hbar\partial/\partial p$ has the same formal expression, it acts on a different space of functions, namely

$$\begin{aligned}
\mathcal{D}(X) = \Big\{\phi, \phi' \in \mathcal{L}^2\left(\frac{-\pi}{2\sqrt{\beta}}, \frac{+\pi}{2\sqrt{\beta}}\right);\, &\phi\left(\frac{+\pi}{2\sqrt{\beta}}\right) \\
&= \phi\left(\frac{-\pi}{2\sqrt{\beta}}\right) = 0\Big\},
\end{aligned} \tag{18}$$

$$\begin{aligned}
\mathcal{D}(X^\dagger) = \Big\{\psi, \psi' \in \mathcal{L}^2\left(\frac{-\pi}{2\sqrt{\beta}}, \frac{+\pi}{2\sqrt{\beta}}\right);& \\
\text{no other restriction on } \psi\Big\}.&
\end{aligned} \tag{19}$$

As it is also shown in Ref. [27], any operator X which obeys the uncertainty relation (1) is merely symmetric. On the other hand, since there are no Dirichlet boundary conditions on the wave functions in the position space ($-\infty < x < \infty$), P is still self-adjoint. In the next section and after finding the momentum eigenfunctions, we prove the self-adjointness property of P using von Neumann's theorem.

To proceed further, let us consider the following Hamiltonian

$$H = \frac{P^2}{2m} + V(X), \tag{20}$$

which using Eqs. (8) and (9) can be written exactly and also perturbatively as

$$H = \frac{\tan^2\left(\sqrt{\beta}p\right)}{2\beta m} + V(x), \tag{21}$$

$$= H_0 + \sum_{n=3}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)(2n-1)B_{2n}}{2m(2n)!}\beta^{n-2}p^{2(n-1)}, \tag{22}$$

where $H_0 = p^2/2m + V(x)$ and B_n is the nth Bernoulli number. So the corrected terms in the modified Hamiltonian are only momentum dependent and proportional to $p^{2(n-1)}$ for $n \geq 3$. As we shall explicitly show, the presence of these terms leads to a positive shift in the particle's energy spectrum. Note that, in general, even for the self-adjoint position and momentum operators, it is by no means obvious that the resulting Hamiltonian will be self-adjoint until and unless the potential term is specified and the appropriate domain is chosen. It is worth to mention that all our calculations are in one-dimensional space. Indeed, in higher dimensions it is necessary to have noncommutativity of coordinates in order to satisfy the Jacobi identity as done by KMM [25]. In one-dimensional space, the Jacobi identity is automatically satisfied. Also, one may relax the point size property of the particle as in the string theory. So we can interpret Eq. (21) as the Schrödinger equation for the particle with size $\sim \hbar\sqrt{\beta}$, where the effect of the nonzero size effectively appears in the kinetic part of the Hamiltonian.

In the quantum domain, this Hamiltonian results in the following generalized Schrödinger equation in the quasi-

position representation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2}+\sum_{n=3}^{\infty}\alpha_n\hbar^{2(n-1)}\beta^{n-2}\frac{\partial^{2(n-1)}\psi(x)}{\partial x^{2(n-1)}} +V(x)\psi(x)=E\,\psi(x), \tag{23}$$

where $\alpha_n = 2^{2n}(2^{2n}-1)(2n-1)B_{2n}/2m(2n)!$ and the second term is due to the GUP corrected terms in (22).

Among infinite possible canonical transformations (CTs), our proposal (8) and (9) has some useful and novel properties. First, it does not change the nature of the position operator and, consequently, the potential term and only modifies the momentum or the kinetic operator. So, among several CTs, only this one preserves the ordinary nature of the position operator. Second, this formalism lets us to write the Hamiltonian as $H = H_0+\beta H_1+\beta^2 H_2+...$, where $H_0 = p^2/2m+V(x)$ is the ordinary Hamiltonian and $H_1, H_2, ...$ contain only the momentum operator. So, using the perturbation theory, the unperturbed eigenfunctions satisfy $H_0|\psi_0\rangle = E_0|\psi_0\rangle$ and we can find $\langle H_1\rangle, \langle H_2\rangle, ...$ in an straightforward manner as done for various cases such as Ref. [26]. In other CTs like the KMM proposal, we cannot decompose the Hamiltonian in such configurations. So, in this sense, this proposal is compatible with perturbative representations. Third, this proposal predicts the existence of a maximal canonical momentum. In fact, the particular form of the kinetic part of the Hamiltonian (21) implies the existence of a maximal momentum:

$$p_{max} = \frac{\pi}{2\sqrt{\beta}} = \frac{\pi M_{Pl}c}{2\sqrt{\beta_0}}, \tag{24}$$

which mimics the recent GUP proposal predicting the presence of both a minimal length uncertainty and a maximal momentum uncertainty through a doubly special relativity consideration [30–33]. There, the generalized momentum has an upper bound proportional to $M_{Pl}c/\alpha_0$ where α_0 similar to β_0 is of the order of unity. However, for our case, the generalized momentum P has no upper bound and it is not physically equivalent with aforementioned GUP. Therefore, the idea of a maximum "canonical" momentum naturally arises from our representation.

III. GUP AND THE FREE PARTICLE

In ordinary quantum mechanics, the free particle wave function $u_p(x)$ is defined as the eigenfunction of the momentum operator P_{op}

$$P_{op}u_p(x) = p\,u_p(x), \tag{25}$$

where p is the eigenvalue. The momentum operator has the following representation in the quasiposition space

$$P_{op} = \frac{\hbar}{i}\frac{\partial}{\partial x}. \tag{26}$$

So, from Eq. (25) we have

$$\frac{\hbar}{i}\frac{\partial u_p(x)}{\partial x} = p u_p(x), \tag{27}$$

which has the following solution

$$u_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ipx}{\hbar}\right), \tag{28}$$

where the constant of integration is chosen to satisfy

$$\int_{-\infty}^{\infty} u_p^*(x)u_p(x')\mathrm{d}p = \delta(x-x'). \tag{29}$$

In GUP scenario, to find the momentum eigenfunction in the position space, we write the momentum operator (9) as $P_{op} = \tan\left(-i\hbar\sqrt{\beta}\partial_x\right)/\sqrt{\beta}$ which results in the following eigenvalue equation

$$\frac{\tan\left(-i\hbar\sqrt{\beta}\partial_x\right)}{\sqrt{\beta}}u_p(x) = p u_p(x). \tag{30}$$

Now, let us consider a class of solutions which satisfies Eqs. (27) and (30) at the same time, but with different eigenvalues [$p \to p'$ in Eq. (27)], i.e.,

$$u_p(x) = A(p)\exp\left(\frac{ip'x}{\hbar}\right), \tag{31}$$

where $p' = f(p)$. Inserting this solution in Eq. (30) results in $\tan\left(\sqrt{\beta}p'\right)/\sqrt{\beta} = p$ or

$$p' = \frac{1}{\sqrt{\beta}}\arctan\left(\sqrt{\beta}p\right), \tag{32}$$

so we have

$$u_p(x) = A(p)\exp\left[\frac{i}{\hbar\sqrt{\beta}}\arctan\left(\sqrt{\beta}p\right)x\right]. \tag{33}$$

To obtain $A(p)$, we demand that the momentum eigenfunction satisfies Eq. (23) of Ref. [25] as the modified version of (29) which results in

$$A(p) = \left[2\pi\hbar\left(1+\beta p^2\right)\right]^{-1/2}. \tag{34}$$

Thus, we finally obtain the momentum eigenfunctions as

$$u_p(x) = \frac{1}{\sqrt{2\pi\hbar\left(1+\beta p^2\right)}}\exp\left[\frac{i}{\hbar\sqrt{\beta}}\arctan\left(\sqrt{\beta}p\right)x\right], \tag{35}$$

which, to the first order agrees with the solution presented in Ref. [22] i.e.

$$u_p(x) = \left(\frac{1-\beta p^2}{2\pi\hbar}\right)^{1/2}\exp\left[\frac{i}{\hbar}\left(p-\frac{\beta}{3}p^3\right)x\right]. \tag{36}$$

Note that this solution for $\beta \to 0$ reduces to (28) in order to satisfy the correspondence principle. Moreover, this result is similar to the position eigenvectors obtained by KMM, where they used (3) and (4) subjected to the deformed scalar product (5). In fact, the

factor $1/\left(1+\beta p^2\right)$ in the definition of the scalar product (5) indeed appeared in the momentum-dependent normalization coefficient of the momentum eigenfunctions, namely,

$$|A(p)|^2 \sim \frac{1}{1+\beta p^2}. \tag{37}$$

At this point, we can use the following theorem to check the self-adjointness property of the position and momentum operators [28, 29]

Theorem 1. (von Neumann's theorem) *For an operator A with deficiency indices (n_+, n_-) there are three possibilities:*

1. If $n_+ = n_- = 0$, then A is self-adjoint (this is a necessary and sufficient condition).

2. If $n_+ = n_- = n \geq 1$, then A has infinitely many self-adjoint extensions, parameterized by a unitary $n \times n$ matrix.

3. If $n_+ \neq n_-$, then A has no self-adjoint extension.

To use von Neumann's theorem, we have to find the wave functions $\phi_\pm$ given by

$$P^\dagger \phi_\pm(x) = \frac{\tan\left(-i\hbar\sqrt{\beta}\partial_x\right)}{\sqrt{\beta}}\phi_\pm(x) = \pm i\lambda\phi_\pm(x). \tag{38}$$

So using Eq. (33) we have

$$\phi_\pm(x) = C_\pm \exp\left[\frac{\mp 1}{\hbar\sqrt{\beta}}\tanh^{-1}\left(\sqrt{\beta}\lambda\right)x\right]. \tag{39}$$

Since the operator P is defined on the whole real axis where $\phi_\pm$ diverge at $x \to \mp\infty$ and consequently are not normalizable, none of the functions $\phi_\pm$ belong to the Hilbert space $\mathcal{L}^2(\mathbb{R})$ and therefore the deficiency indices are $(0,0)$. Hence, we conclude that the momentum operator is indeed self-adjoint with the following domain

$$\mathcal{D}(P) = \mathcal{D}(P^\dagger) = \{\phi \in \mathcal{D}_{max}(\mathbb{R})\}, \tag{40}$$

where $\mathcal{D}_{max}$ denotes the maximal domain on which the operator P has a well defined action, i.e., $\mathcal{D}_{max}(P) = \left\{\phi \in \mathcal{L}^2(\mathbb{R}) : P\phi \in \mathcal{L}^2(\mathbb{R})\right\}$. Using the same procedure for the position operator X on the finite interval, it is straightforward to check that both $\phi_\pm(p) = C_\pm e^{\mp\lambda p}$ belong to $\mathcal{L}^2(-\frac{1}{2}\pi\beta^{-1/2}, \frac{1}{2}\pi\beta^{-1/2})$ and the deficiency indices are $(1,1)$. Therefore, one concludes that the position operator is no longer essentially self-adjoint but has a one-parameter family of self-adjoint extensions which is in agreement with the previous result.

IV. GUP AND THE HARMONIC OSCILLATOR

In this section, we study the classical and quantum mechanical solutions of the harmonic oscillator in the GUP framework and present its semiclassical results. Moreover, we study the effects of the minimal length uncertainty on the thermodynamic aspects of the harmonic oscillator in both classical and quantum domains.

A. Classical Description

Let us consider the Hamiltonian of a particle of mass m confined in a quadratic potential

$$H^{(HO)} = \frac{\tan^2\left(\sqrt{\beta}p\right)}{2\beta m} + \frac{1}{2}m\omega^2 x^2, \tag{41}$$

which using the Hamiltonian equations results in

$$\dot{x} = \frac{\tan(\sqrt{\beta}p)\sec^2(\sqrt{\beta}p)}{\sqrt{\beta}m}, \tag{42}$$

$$\dot{p} = -m\omega^2 x. \tag{43}$$

So, in the GUP formalism, the velocity $\dot{x}$ is not equal to p/m, but it tends to p/m as β goes to zero. Using Eqs. (42) and (43) we obtain

$$\ddot{p} + \omega^2\frac{\tan(\sqrt{\beta}p)\sec^2(\sqrt{\beta}p)}{\sqrt{\beta}} = 0. \tag{44}$$

If we set the initial conditions as $x(0) = a$ and $p(0) = 0$, it is straightforward to check that the above equation admits the following solutions

$$p(t) = \pm\frac{1}{\sqrt{\beta}}\arctan\left(\frac{\eta}{\sqrt{1+(1+\eta^2)\cot^2\left(\sqrt{1+\eta^2}\omega t\right)}}\right), \tag{45}$$

$$x(t) = \mp\frac{a\sqrt{1+\eta^2}\cot\left(\sqrt{1+\eta^2}\omega t\right)}{\sqrt{1+(1+\eta^2)\cot^2\left(\sqrt{1+\eta^2}\omega t\right)}}, \tag{46}$$

where $\eta = \sqrt{\beta}m\omega a$. So the actual frequency of the harmonic oscillator in GUP scenario increases with respect to the absence of GUP as $\bar{\omega} = \sqrt{1+\beta m^2\omega^2 a^2}\,\omega \geq \omega$. In fact, this frequency depends on GUP parameter, particle's mass, and the initial position. Moreover, as β increases, the particle is often located at the end points $\pm a$ and the accessible phase space decreases with respect to the absence of GUP (see Fig. 1).

B. Semiclassical description

Before studying the corresponding generalized Schrödinger equation, it is worthwhile to find the quantized energy spectrum using the semiclassical scheme. The Wentzel-Kramers-Brillouin (WKB) quantization rule, represented succinctly by the formula

$$\oint p\,\mathrm{d}q = \left(n+\frac{1}{2}\right)h, \qquad n = 0, 1, \ldots, \tag{47}$$

allows us to find the approximate energy spectrum and in ordinary quantum mechanics gives the exact results. Using Eq. (41) we find

$$\begin{aligned}\oint p\,\mathrm{d}x &= \frac{2}{\sqrt{\beta}}\int_{-a}^{a}\arctan\left(\sqrt{\beta}m\omega\sqrt{a^2-x^2}\right)\mathrm{d}x \\ &= 2\pi\frac{\sqrt{1+\beta m^2\omega^2 a^2}-1}{\beta m\omega},\end{aligned} \tag{48}$$

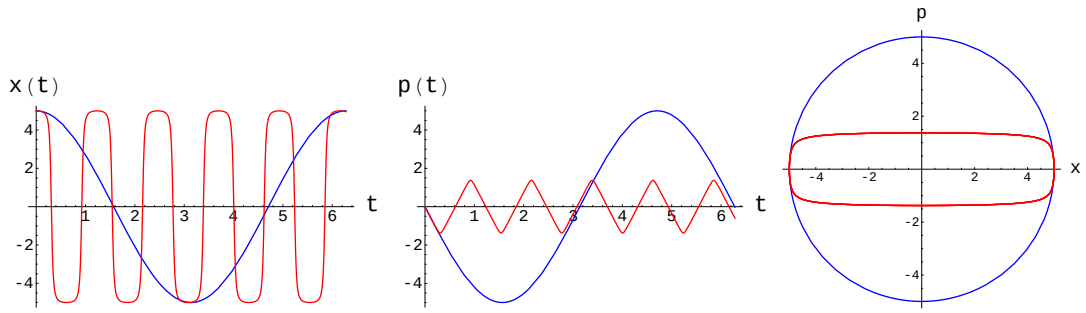

FIG. 1. The temporal behavior of x and p, and the phase space of the harmonic oscillator for $\beta = 0$ (blue line) and $\beta = 1$ (red line). We set $m = \omega = 1$ and $a = 5$ ($\bar{\omega} = \sqrt{26}$).

which results in the following semiclassical energy spectrum

$$E_n^{(SC)} = \frac{1}{2}m\omega^2 a_n^2, \\ = -\frac{1}{8}\gamma\hbar\omega + \hbar\omega\left(n + \frac{1}{2}\right)(1 + \gamma/2) + \frac{1}{2}\hbar\omega\gamma n^2, \quad (49)$$

where $\gamma = \beta m\hbar\omega$. As we have expected, $E_n^{(SC)}$ tends to $\hbar\omega\,(n + 1/2)$ as β goes to zero. However, contrary to the ordinary formulation where $E_n^{(SC)}$ is equal to the exact energy spectrum, it does not give the exact spectrum in the GUP formalism. This is due to the fact that from the Hamiltonian (41) we expect that the energy spectrum depends on numerous powers of β, but $E_n^{(SC)}$ only represents a linear dependence of the GUP parameter. In the next section, we show this fact by a rigorous mathematical proof. However, it can be considered as a good approximation which is related to the correct quadratic dependence on the quantum number.

C. Quantum Description

For the case of the harmonic oscillator, because of the quadratic form of the potential $V(x) = 1/2\, m\omega^2 x^2$, we obtain a second-order differential equation in the momentum space, namely

$$-\frac{\partial^2\phi(p)}{\partial p^2} + \frac{\tan^2\left(\sqrt{\gamma}p\right)}{\gamma}\phi(p) = \bar{\epsilon}\,\phi(p), \quad (50)$$

where $p \to \sqrt{m\hbar\omega}\,p$, $\gamma = m\hbar\omega\beta$, and $\bar{\epsilon} = \frac{2E}{\hbar\omega}$. In terms of the new variable $z = \sqrt{\gamma}p$, we obtain

$$\left[-\frac{\partial^2}{\partial z^2} + \nu(\nu-1)\tan^2(z) - \epsilon(\nu)\right]\phi(z;\nu) = 0, \quad (51)$$

where by definition

$$\nu = \frac{1}{2}\left(1 + \sqrt{1 + \frac{4}{\gamma^2}}\right), \qquad \epsilon(\nu) = \frac{\bar{\epsilon}}{\gamma}, \quad (52)$$

and the boundary condition is

$$\phi(z;\nu)\Big|_{z=\pm\pi/2} = 0. \quad (53)$$

The above differential equation is exactly solvable and the eigenfunctions can be obtained in terms of Gauss hypergeometric functions where we briefly present the solutions [34].

To find the even parity states, let us use the substitution $\xi = \sin^2(z)$ which leads to

$$\xi(1-\xi)\frac{\partial^2\phi(\xi;\nu)}{\partial\xi^2} + \left(\frac{1}{2} - \xi\right)\frac{\partial\phi(\xi;\nu)}{\partial\xi} \\ + \left[\Delta(\nu,\epsilon) - \frac{1}{4}\frac{\nu(\nu-1)}{1-\xi}\right]\phi(\xi;\nu) = 0, \quad (54)$$

where $\Delta(\nu,\epsilon) = \frac{1}{4}[\nu(\nu-1) + \epsilon(\nu)]$. Now, to get rid of the regular singularity of the last term we search for the solution of the form

$$\phi(\xi;\nu) = (1-\xi)^a\, Y(\xi;\nu), \quad (55)$$

where a satisfy the algebraic equation

$$a^2 - \frac{1}{2}a - \frac{1}{4}\nu(\nu-1) = 0. \quad (56)$$

So we obtain the Gauss hypergeometric equation for the variable $Y(\xi;\nu)$

$$\xi(1-\xi)Y'' + \left(\frac{1}{2} - (\alpha+\beta+1)\xi\right)Y' - \alpha\beta Y = 0 \quad (57)$$

subjected to $\alpha + \beta = 2a$ and $\alpha\beta = a^2 - \Delta(\nu,\epsilon)$. This equation admits two independent solutions. However, the physically acceptable solution which vanishes at the boundary $\lim_{\xi\to 1} Y(\xi;\nu) = 0$ is

$$Y(\xi;\nu) = \mathcal{A}(\nu)(1-\xi)^{\nu/2}\,{}_2F_1\left(\alpha,\beta;\nu+\frac{1}{2};1-\xi\right) \quad (58)$$

where $\mathcal{A}(\nu)$ is the normalization constant. The analyticity and the convergence of the hypergeometric function for all $\xi \in [0,1]$ results in

$$\alpha \quad \text{or} \quad \beta = -k, \qquad k = 0,1,2,\ldots. \quad (59)$$

So we obtain the even parity eigenfunctions

$$Y_{2k}(\xi;\nu) = \mathcal{A}_k(\nu)(1-\xi)^{\nu/2}\,{}_2F_1\left(-k,\nu+k;\nu+\frac{1}{2};1-\xi\right), \quad (60)$$

and the eigenvalues

$$\epsilon_{2k}(\nu) = 4k(\nu+k)+\nu, \qquad k=0,1,2,\ldots. \quad (61)$$

Finally, in terms of the original variable p we have

$$\phi_{2k}(p;\gamma) = \mathcal{A}_k(\nu)\left[\cos(\sqrt{\gamma}p)\right]^{\left(1+\sqrt{1+\frac{4}{\gamma^2}}\right)/2} \times {}_2F_1\left(-k,\nu+k;\nu+\frac{1}{2};\cos^2(\sqrt{\gamma}p)\right). \quad (62)$$

To find the antisymmetric solutions let us define

$$\phi(z;\nu) = \sin(z)\varphi(z;\nu), \quad (63)$$

where ϕ is an even function of z. By substitution of this solutions in the original equation we have

$$\left[-\frac{\partial^2}{\partial z^2} - 2\cot(x)\frac{\partial}{\partial z} + \nu(\nu-1)\tan^2(z) + 1 - \epsilon(\nu)\right] \times\varphi(z;\nu) = 0, \quad (64)$$

where by choosing $\xi = \sin^2(z)$ can be written as

$$\left[\xi(1-\xi)\frac{\partial^2}{\partial\xi^2} + \left(\frac{3}{2}-2\xi\right)\frac{\partial}{\partial\xi} + \Delta(\nu,\epsilon) - \frac{1}{4} - \frac{1}{4}\frac{\nu(\nu-1)}{1-\xi}\right]\varphi(\xi;\nu) = 0. \quad (65)$$

Similar to the procedure for the even states let us define

$$\phi(\xi;\nu) = (1-\xi)^{\nu/2}\,U(\xi;\nu), \quad (66)$$

which converts Eq. (65) to the Gauss hypergeometric equation

$$\xi(1-\xi)U'' + \left(\frac{3}{2} - (\bar{\alpha}+\bar{\beta}+1)\xi\right)U' - \bar{\alpha}\bar{\beta}U = 0, \quad (67)$$

where $\bar{\alpha} = \frac{1}{2}(\nu+1) - \sqrt{\Delta(\nu,\epsilon)}$ and $\bar{\beta} = \frac{1}{2}(\nu+1) + \sqrt{\Delta(\nu,\epsilon)}$. As before we set $\bar{\alpha} = -k$ and find the eigenenergies

$$\epsilon_{2k+1}(\nu) = (2k+1)(2\nu+2k+1)+\nu, \qquad k=0,1,\ldots, \quad (68)$$

for the antisymmetric eigenfunctions

$$U_{2k+1}(\xi;\nu) = \mathcal{B}_k(\nu)\sqrt{\xi}(1-\xi)^{\nu/2} \times {}_2F_1\left(-k,\nu+k+1;\nu+\frac{1}{2};1-\xi\right). \quad (69)$$

In terms of the original variables we have

$$\phi_{2k+1}(p;\gamma) = \mathcal{B}_k(\nu)\sin(\sqrt{\gamma}p)\left[\cos(\sqrt{\gamma}p)\right]^{\left(1+\sqrt{1+\frac{4}{\gamma^2}}\right)/2} \times {}_2F_1\left(-k,\nu+k+1;\nu+\frac{1}{2};\cos^2(\sqrt{\gamma}p)\right). \quad (70)$$

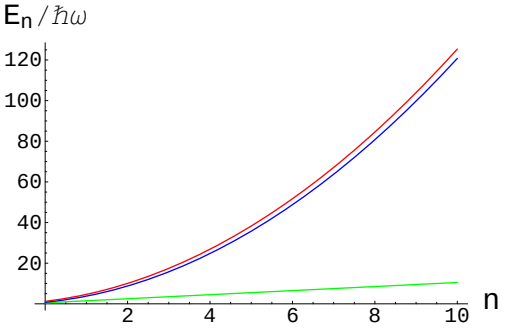

FIG. 2. Comparing $E_n/\hbar\omega$ (red line) and $E_n^{(SC)}/\hbar\omega$ (blue line) for $\gamma = 2$ with the ordinary harmonic oscillator spectrum (green line).

Note that we can combine Eqs. (61) and (68) in a single formula to express the full spectrum, namely $\epsilon_n(\nu) = n(2\nu+n)+\nu$ for $n = 0,1,2,\ldots$ or

$$E_n(\gamma) = \hbar\omega\left(n+\frac{1}{2}\right)\left(\sqrt{1+\gamma^2/4}+\gamma/2\right) + \frac{1}{2}\hbar\omega\gamma n^2, \quad (71)$$

in terms of γ. So, as we have expected, this result exactly coincides with the spectrum of the harmonic oscillator in the formalism proposed by Kempf, Mangano, and Mann. In Fig. 2, we have depicted the energy spectrum of the harmonic oscillator in GUP framework (71), its semiclassical approximation (49), and its spectrum in ordinary quantum mechanics. The efficiency of the semiclassical solution is manifest in the figure. In fact, to first order of the GUP parameter, E_n is equal to $E_n^{(SC)}$ up to a positive constant, namely

$$E_n \simeq E_n^{(SC)} + \frac{1}{8}\gamma\hbar\omega. \quad (72)$$

To check the self-adjointness property of $H^{(HO)}$, it is natural to present the sesquilinear form for ψ and ϕ as

$$\begin{aligned} 2\,iB(\psi,\phi) &= \langle H\psi|\phi\rangle - \langle\psi|H|\phi\rangle, \\ &= \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \mathrm{d}p\ (H\psi(p))^*\,\phi(p) - \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \mathrm{d}p\,\psi^*(p)H\phi(p), \\ &= -\frac{1}{2}m\omega^2\hbar^2\left[\int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \mathrm{d}p\,\psi''(p)^*\phi(p) - \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} \mathrm{d}p\,\psi^*(p)\phi''(p)\right], \\ &= \frac{1}{2}m\omega^2\hbar^2\left[\psi^*(p)\phi'(p)\Big|_{p=\frac{\pi}{2\sqrt{\beta}}} - \psi'^*(p)\phi(p)\Big|_{p=\frac{\pi}{2\sqrt{\beta}}}\right. \\ &\quad \left. -\psi^*(p)\phi'(p)\Big|_{p=\frac{-\pi}{2\sqrt{\beta}}} + \psi'^*(p)\phi(p)\Big|_{p=\frac{-\pi}{2\sqrt{\beta}}}\right]. \end{aligned} \quad (73)$$

On the other hand, using the explicit form of the solutions (62) and (70), it is straightforward to check that the first derivative of the solutions as well as $\phi(p;\nu)$ vanishes at the boundaries i.e.

$$\phi'(p;\nu)\Big|_{p=\frac{\pm\pi}{2\sqrt{\beta}}} = 0. \quad (74)$$

Therefore, $\psi^*(p)$ and $\psi'^*(p)$ can take arbitrary values at the boundaries. This means that the domain of the adjoint of H is larger than that of H, so the Hamiltonian is symmetric but not self-adjoint. The domains are

$$\mathcal{D}(H) = \left\{\phi \in \mathcal{D}_{max}\left(\frac{-\pi}{2\sqrt{\beta}}, \frac{+\pi}{2\sqrt{\beta}}\right); \phi\left(\frac{+\pi}{2\sqrt{\beta}}\right) = \phi\left(\frac{-\pi}{2\sqrt{\beta}}\right) = \phi'\left(\frac{+\pi}{2\sqrt{\beta}}\right) = \phi'\left(\frac{-\pi}{2\sqrt{\beta}}\right) = 0\right\}, \quad (75)$$

$$\mathcal{D}(H^\dagger) = \left\{\psi \in \mathcal{D}_{max}\left(\frac{-\pi}{2\sqrt{\beta}}, \frac{+\pi}{2\sqrt{\beta}}\right); \text{no other restriction on } \psi\right\}. \quad (76)$$

Note that this result is not surprising because even the Hamiltonian of the one-dimensional particle in a box is not a truly self-adjoint operator as well [29].

D. Classical partition function

In statistical mechanics, the canonical partition function of N identical, one-dimensional oscillators which encodes the statistical properties of a thermodynamic system can be written in the classical domain as

$$Z(b) = \frac{1}{N!h^N}\int \exp\left[-bH(p_1\cdots p_N, x_1\cdots x_N)\right] \times \mathrm{d}p_1\cdots\mathrm{d}p_N\,\mathrm{d}x_1\cdots\mathrm{d}x_N, \quad (77)$$

where $b \equiv 1/k_BT$, k_B denotes Boltzmann's constant and T is the temperature. For N noninteracting oscillators, the total partition function can be obtained from the single-particle partition function as

$$Z(b) = \frac{1}{N!h^N}\left[\int_{-\infty}^{+\infty}\exp\left[-\frac{1}{2}bm\omega^2x^2\right]\mathrm{d}x\right]^N \times \left[\int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}}\exp\left[-\frac{b}{2\beta m}\tan^2(\sqrt{\beta}p)\right]\mathrm{d}p\right]^N \quad (78)$$

$$= \frac{1}{N!h^N}\left(\frac{2\pi k_BT}{m\omega^2}\right)^{N/2} \times \left(\frac{\pi\exp\left(\frac{1}{2\beta mk_BT}\right)\mathrm{erfc}\left(\frac{1}{\sqrt{2\beta mk_BT}}\right)}{\sqrt{\beta}}\right)^N, \quad (79)$$

where $\mathrm{erfc}(x)$ is the complementary error function. Using the asymptotic expansion of the complementary error function for large x, namely

$$\mathrm{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}}\left[1 + \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot3\cdot5\cdots(2n-1)}{(2x^2)^n}\right] \quad (80)$$

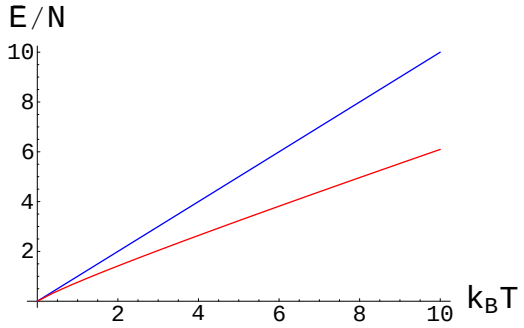

FIG. 3. The classical mean energy of N harmonic oscillators in thermal equilibrium versus temperature for $\beta = 1$ (red line) and $\beta = 0$ (blue line). We set $m = 1$.

we can write the partition function in terms of powers of k_BT as

$$Z(b) = \frac{1}{N!}\left(\frac{k_BT}{\hbar\omega}\right)^N\left(1 + \sum_{n=1}^{\infty}(2n-1)!!\,(-\beta mk_BT)^n\right)^N, \quad (81)$$

where for $\beta \to 0$ reduces to the ordinary partition function. Also, the classical mean energy of the system is given by

$$\overline{E}_C = -\frac{\partial}{\partial b}\ln Z = N\left(\frac{k_BT}{2} + \sqrt{\frac{k_BT}{2\pi\beta m}}\frac{\exp\left(-1/2\beta mk_BT\right)}{\mathrm{erfc}\left(\sqrt{1/2\beta mk_BT}\right)} - \frac{1}{2\beta m}\right), \quad (82)$$

$$= N\left(k_BT + \frac{\sum_{n=1}^{\infty}n(2n-1)!!\,(-\beta m)^n\,(k_BT)^{n+1}}{1 + \sum_{n=1}^{\infty}(2n-1)!!\,(-\beta mk_BT)^n}\right), \quad (83)$$

which goes to Nk_BT for $\beta \to 0$. Therefore, as indicated in Fig. 3, in the presence of GUP, the mean energy decreases with respect to $\beta = 0$. The reason for the reduction of mean energy with respect to $\beta = 0$ is a consequence of the reduction of phase space volume (surface) due to possible definition of a rescaled $\hbar$. In fact the volume of the fundamental cell increases in the presence of the minimal length uncertainty relation and the number of degrees of freedom reduces consequently. Moreover, it modifies the Helmholtz free energy $A = -k_BT\ln Z$ and the entropy $S = k_B\ln Z + \overline{E}/T$ as well. The above equation shows that the equipartition theorem fails in the GUP scenario. Although the averaged potential satisfies the equipartition theorem i.e. $\langle 1/2m\omega^2x^2\rangle = k_BT/2$, the kinetic part yields the smaller value $\langle K\rangle < k_BT/2$ [see Eq. (83)]. Similarly, the heat capacity at constant volume which is proportional to $\frac{\partial\overline{E}}{\partial T}$, decreases with respect to the absence of GUP, namely

$$C_V^{\beta\neq0} < C_V^{\beta=0}. \quad (84)$$

Note that, for the case of the ideal gas we can write the

partition function as

$$Z(b)=\frac{V^N}{N!h^N}\left(\frac{\pi\exp\left(1/2\beta mk_BT\right)\mathrm{erfc}\left(\sqrt{1/2\beta mk_BT}\right)}{\sqrt{\beta}}\right)^N \quad (85)$$

so, using the definition of pressure $P=\frac{1}{b}\frac{\partial \ln Z}{\partial V}$, we recover the ordinary ideal gas equation of state $PV=Nk_BT$. However, the corresponding heat capacity will be modified as mentioned above.

E. Quantum partition function

In the quantum statistical mechanics, the partition function for a single oscillator is given by

$$Z(b)=\sum_{n=0}^{\infty}\exp\left(-bE_n\right), \quad (86)$$

where the energy eigenvalues are defined in Eq. (71). Now,

$$Z(b;\gamma)=e^{-(1/2)b\hbar\omega\left(\sqrt{1+\gamma^2/4}+\gamma/2\right)} \times\sum_{n=0}^{\infty}\exp\left[-b\hbar\omega\left(\left(\sqrt{1+\gamma^2/4}+\gamma/2\right)n+\frac{1}{2}\gamma n^2\right)\right] \quad (87)$$

$$=e^{-(1/2)b\hbar\omega\left(\sqrt{1+\gamma^2/4}+\gamma/2\right)}P(b;\gamma), \quad (88)$$

where we defined $P(b;\gamma)\equiv\sum_{n=0}^{\infty}\exp\left[-b\hbar\omega\left(\left(\sqrt{1+\gamma^2/4}+\gamma/2\right)n+\frac{1}{2}\gamma n^2\right)\right]$. So we have $P(b;0)=\frac{1}{1-\exp(-b\hbar\omega)}$ and $Z(b;0)=\frac{\exp(-(1/2)b\hbar\omega)}{1-\exp(-b\hbar\omega)}$. Also, the mean energy of the oscillator is given by

$$\overline{E}=-\frac{\partial}{\partial b}\ln Z=\frac{1}{2}\hbar\omega\left(\sqrt{1+\gamma^2/4}+\gamma/2\right)-\frac{P'(b;\gamma)}{P(b;\gamma)}, \quad (89)$$

$$=\hbar\omega\left(\sqrt{1+\gamma^2/4}+\gamma/2\right) \times\left(\frac{1}{2}+\frac{\sum_{n=0}^{\infty}\left(n+\frac{n^2}{1+\sqrt{1+4/\gamma^2}}\right)e^{-\frac{\hbar\omega}{k_BT}\left(\left(\sqrt{1+\gamma^2/4}+\gamma/2\right)n+\frac{1}{2}\gamma n^2\right)}}{\sum_{n=0}^{\infty}e^{-\frac{\hbar\omega}{k_BT}\left(\left(\sqrt{1+\gamma^2/4}+\gamma/2\right)n+\frac{1}{2}\gamma n^2\right)}}\right), \quad (90)$$

where prime denotes the derivative with respect to b. The mean energy of the harmonic oscillator in the quantum domain and in the GUP formalism is depicted in Fig. 4 which shows a modified minimum value in the low temperature limit

$$\overline{E}\simeq\frac{1}{2}\hbar\omega\left(\sqrt{1+\gamma^2/4}+\gamma/2\right). \quad (91)$$

To compare the classical and quantum results in the high-temperature limit, we can write Eq. (82) as

$$\frac{\overline{E}_C}{N\hbar\omega}=\frac{1}{2}\left(\frac{k_BT}{\hbar\omega}+\sqrt{\frac{k_BT/\hbar\omega}{2\pi\gamma}}\frac{\exp\left(\frac{-1/2\gamma}{k_BT/\hbar\omega}\right)}{\mathrm{erfc}\left(\sqrt{\frac{1/2\gamma}{k_BT/\hbar\omega}}\right)}-\frac{1}{\gamma}\right). \quad (92)$$

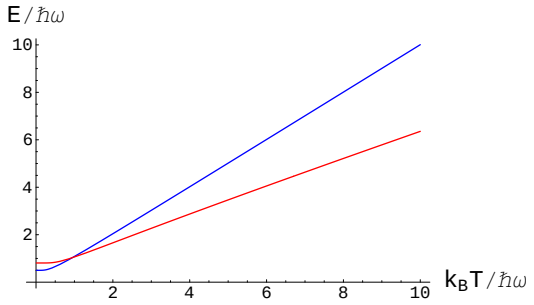

FIG. 4. The quantum mechanical mean energy of the harmonic oscillator $E_n/\hbar\omega$ versus $k_BT/\hbar\omega$ for $\gamma=1$ (red line) and $\gamma=0$ (blue line).

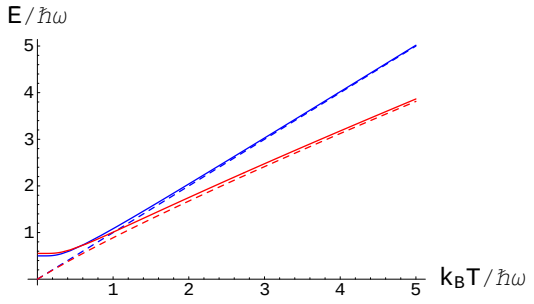

FIG. 5. The classical (dashed line) and quantum mechanical (solid line) mean energy of the harmonic oscillator for $\gamma=0.2$ (red line) and $\gamma=0$ (blue line).

In Fig. 5, the classical (92) and the quantum mechanical (90) mean energy of the harmonic oscillator for $\gamma=0.2$ is depicted and compared with the ordinary thermodynamic results.

V. GUP AND THE POTENTIALS WITH SHARP BOUNDARIES

In the GUP scenario, we cannot measure the position of a particle with an uncertainty less than $(\Delta X)_{min}$. So, in principle, it is not possible to properly define the potentials with infinitely sharp boundaries (It is well known that such sharp boundaries cannot be also defined in theories with space-time uncertainty [35, 36]). Indeed, the position of the boundaries can be only determined within this uncertainty. However, one may argue that in a first-step analysis, the assumption of sharp boundaries would be an acceptable approximation. But the validity of this approximation requires that the uncertainty in the energy spectrum due to the boundaries' position uncertainty to be much smaller than the GUP energy correction.

To investigate this point, we study the problem of a

particle in a box as an example of a potential with infinitely sharp boundaries and compare both energy corrections. So, let us consider a particle with mass m confined in an infinite one-dimensional box with length L

$$V(x)=\begin{cases}0 & 0<x<L,\\ \infty & \text{elsewhere.}\end{cases} \tag{93}$$

The corresponding eigenfunctions should satisfy the following generalized Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_n(x)}{\partial x^2}+\sum_{j=3}^{\infty}\alpha_j\hbar^{2(j-1)}\beta^{j-2}\frac{\partial^{2(j-1)}\psi(x)}{\partial x^{2(j-1)}} =E_n\,\psi_n(x), \tag{94}$$

for $0<x<L$ and they also meet the boundary conditions $\psi_n(0)=\psi_n(L)=0$. Because of the boundary conditions, the eigenfunctions do not change with respect to the absence of the GUP ($\beta=0$) [22]. This fact leads us to consider the following additional condition for the eigenfunctions

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_n(x)}{\partial x^2}=\varepsilon_n\psi_n(x), \qquad 0<x<L, \tag{95}$$

where $\varepsilon_n=\frac{n^2\pi^2\hbar^2}{2mL^2}$. If this condition is also satisfied, we can write the second term in Eq. (94) in terms of $\psi_n(x)$ i.e.

$$\frac{\partial^{2(j-1)}\psi_n(x)}{\partial x^{2(j-1)}}=\frac{-2m\varepsilon_n}{\hbar^2}\frac{\partial^{2(j-2)}\psi_n(x)}{\partial x^{2(j-2)}}=\cdots =\left(\frac{-2m\varepsilon_n}{\hbar^2}\right)^{j-1}\psi_n(x). \tag{96}$$

So, we have

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_n(x)}{\partial x^2}+\sum_{j=3}^{\infty}\alpha_j\hbar^{2(j-1)}\beta^{j-2}\frac{\partial^{2(j-1)}\psi_n(x)}{\partial x^{2(j-1)}} =\left(\varepsilon_n+\sum_{j=3}^{\infty}|\alpha_j|\beta^{j-2}(2m)^{j-1}\varepsilon_n^{j-1}\right)\psi_n(x). \tag{97}$$

Now, comparing Eqs. (94) and (97) shows that

$$E_n=\frac{\tan^2\left(\sqrt{2m\beta\varepsilon_n}\right)}{2m\beta}, \tag{98}$$

$$=\varepsilon_n+\frac{4}{3}\beta m\varepsilon_n^2+\frac{68}{45}\beta^2m^2\varepsilon_n^3+\frac{496}{315}\beta^3m^3\varepsilon_n^4+\cdots,$$

$$=\frac{\hbar^2}{mL^2}\left[\frac{n^2\pi^2}{2}+\frac{n^4\pi^4}{3}\left(\frac{(\Delta X)_{min}}{L}\right)^2 +\frac{17n^6\pi^6}{90}\left(\frac{(\Delta X)_{min}}{L}\right)^3+\cdots\right]. \tag{99}$$

This GUP corrected energy spectrum can be also obtained using the Wilson-Sommerfeld quantization rule given by

$$\oint p\,\mathrm{d}q=nh, \qquad n=1,2,\ldots, \tag{100}$$

with two conjugate variables p and q and the integer n. Since the potential is constant (zero) inside the box, we have

$$\oint p\,\mathrm{d}x=\frac{2L}{\sqrt{\beta}}\arctan\left(\sqrt{2\beta mE_n}\right). \tag{101}$$

So the semiclassical spectrum is

$$E_n^{(SC)}=\frac{\tan^2\left(\sqrt{\beta}n\pi\hbar/L\right)}{2m\beta}, \tag{102}$$

which exactly coincides with the quantum mechanical spectrum given by Eq. (98). These results show that the GUP energy correction is of order of $\left(\frac{(\Delta X)_{min}}{L}\right)^2$.

Now let us find the energy correction due to the uncertainty in the position of the well's walls

$$\Delta E_n\simeq\left|\frac{d\varepsilon_n}{dL}\right|(\Delta X)_{min}=\frac{n^2\pi^2\hbar^2}{mL^2}\left(\frac{(\Delta X)_{min}}{L}\right), \tag{103}$$

which is first order in $(\Delta X)_{min}/L$. Therefore, the GUP energy correction is much smaller than ΔE_n and cannot be detected in the presence of the minimal length. This result confirms that the particle in a box potential cannot be defined in the GUP framework as in ordinary quantum mechanics. This conclusion can be also generalized to other potentials with infinitely sharp boundaries.

VI. CONCLUSIONS

In this paper, we proposed a nonperturbative gravitational quantum mechanics in agreement with the existence of a minimal length uncertainty relation. In this formalism the generalized Hamiltonian takes the form $H=\tan^2\left(\sqrt{\beta}p\right)/(2\beta m)+V(x)$, where x and p are the ordinary position and momentum operators. We showed that this approach is equivalent with KMM representation and we found the corresponding canonical transformation. This representation has some advantages: First, it modifies only the kinetic part (momentum operator) and the potential term (position operator) remains unchanged. Second, this formalism is compatible with perturbative schemes. Third, this representation predicts the existence of a maximal canonical momentum proportional to $M_{Pl}c/\sqrt{\beta_0}$. Because of the universality of the GUP effects, this formalism can potentially be tested in various quantum mechanical systems, of which we have studied just a few cases.

We thoroughly studied the case of the harmonic oscillator in classical and quantum domains. In the classical domain, we found the trajectory of the oscillating particle and showed that the GUP modified frequency of the oscillator depends on mass, initial position and the GUP parameter. Also, for large β the particle is often located around the end points. In the quantum domain, we obtained the exact energy eigenvalues and the eigenfunctions and showed that they are in agreement with

those obtained in Ref. [25]. Moreover, the quadratic dependence of the energy spectrum on the state number is confirmed using the semiclassical approximation. To address the effects of the generalized uncertainty principle on the thermodynamic properties of the harmonic oscillator, we found the partition functions and the mean energies in both classical and quantum limits. We showed that, in the presence of the GUP and at the fixed temperature, the mean energy and the heat capacity of the oscillator reduce in comparison with those of the ordinary classical and quantum mechanics.

Also, we have indicated that X and $H^{(HO)}$ are merely symmetric, but P is a truly self-adjoint operator. Note that these results for X and P agree with those of KMM representation [25]. However, the difference is that in this representation all these operators are formally self-adjoint, i.e., $A = A^\dagger$ ($A \in \{X, P, H^{(HO)}\}$), but $\mathcal{D}(A) \neq \mathcal{D}(A^\dagger)$ for $A \in \{X, H^{(HO)}\}$ and $\mathcal{D}(P) = \mathcal{D}(P^\dagger)$. On the other hand, in KMM representation only P is formally and truly self-adjoint. The problems with the potentials with sharp boundaries are finally discussed. We showed that for this type of potentials, the GUP energy correction is much smaller than the uncertainty in the energy spectrum due to the boundaries' position uncertainty.

ACKNOWLEDGMENTS

I am very grateful to Kourosh Nozari, Rajesh Parwani and Achim Kempf for fruitful discussions and suggestions and for a critical reading of the manuscript. I would like to thank the referees for giving such constructive comments which considerably improved the quality of the paper.

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